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The theory of the structural identification of non-linear multidimensional systems☆

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ABSTRACT

The qualitative properties of a posteriori dynamic processes, which allow of a continuous Kalman-Mesarovic realization in a class of non-linear time-invariant multidimensional differential systems of minimum dynamic order are investigated when there are no constantly acting preset controls. It is shown that, in the general case, there are structural obstacles on this route. A constructive procedure for obtaining such differential realizations is proposed which is illustrated taking the example of spatial rotational motion (with damping), described by Euler's equations.

The synthesis of a model of the dynamic equations of the system using a priori data on its behaviour (Ref. 1, Definition 1)[1] is considered within the limits of a *realization problem*[2-4] although, until recently, the answer to the question "how is a model constructed ?" was exclusively associated with specific physical laws. Hence, methodologically, the theory of realization can be considered as a first step in constructing a qualitative theory of the structural identification of dynamical systems.[1] In particular, Kalman,[2] stating that "the problem of realization plays a central role in systems theory", formulated the following approach: "We now consider the realization problem of as an attempt to guess the equations of motion of a dynamical system from the behaviour of its input and output signals or as a problem of constructing a physical model which accounts for the experimental data".

As a rule, the parametric identification of dynamical systems, even in a linear formulation, is not accompanied by a discussion of the "adequacy" of the model, and the fact that a model has a linear structure is considered as an a priori datum (for example, see the identification of a model of the dynamics of a nonrigid orbital telescope.[5] On the other hand, the necessary and sufficient conditions for the solvability of the problem of a multidimensional differential realization of an a posteriori model of the dynamics in the class of non-linear time-invariant differential equations of a state with a time-independent output signal matrix are determined and discussed in the methodological context of this paper. Here, the initial system, which is subject to a non-linear realization of minimum dynamic order, is considered using the theoretical-systems concept of a "black box" (Ref. 3, Definition 1.2) as a multiply connected continuous dynamic process obtained experimentally.

1. Formulation of the realization problem and mathematical prerequisites for its solution

Henceforth, R is the field of real numbers, R^q is a q-dimensional Euclidean space over R with a scalar product denoted by $\langle\cdot,\ \cdot\rangle_{R^q}$, $M_{n,m}(R)$ is the space of all $n\times m$ matrices with elements from R, and $T := [t_0, t_1]$ is a segment of the number line R. As usual, $C^\infty(T, R^q)$ is the space of the functions which are infinitely differentiable in T with values in R^q and we assume that it is endowed with the structure of a Euclidean space with a scalar product

$$\langle\varphi, \phi\rangle_{C^\infty} := \int_T \langle\varphi(\tau), \phi(\tau)\rangle_{R^q} d\tau,\ \varphi,\quad \phi\in C^\infty(T, R^q)$$

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We now separate out the class of systems for which the motion in the state space R^n is described by the vector-matrix differential equation

$$dx(t)/dt = Ax(t) + B\psi(x(t)), \quad x(t_0) = x_0 \in R^n, \quad t \in T; \quad y(t) = Cx(t)$$
$$A \in M_{n,n}(R), \quad B \in M_{n,m}(R), \quad C \in M_{p,n}(R), \quad p \le n \tag{1.1}$$

Here, $x \in C^\infty(T, R^n)$ is the trajectory of the system, $\psi(x) \in C^\infty(T, R^m)$ is the non-linear component of the equations of state of the system (or a control in the form of a non-linear feedback) and $y \in C^\infty(T, R^p)$ is the "output" of the system.

We shall consider the differential realization problem (taking account of the second equation in (1.1)) with the limits of the following structural constraint

$$\psi(x(t)) = \varphi(y(t)), \quad t \in T \tag{1.2}$$

which is due to the fact that the hypothesis of a law $\psi(x)$ can be experimentally confirmed or refuted exclusively using the output signal $y(t)$.

1.1. Formulation of the realization problem

For an a posteriori process $y \in C^\infty(T, R^p)$ and for an a priori specified law (as a hypothesis) φ: $C^\infty(T, R^p) \to C^\infty(T, R^m)$ such that

$$\dim \operatorname{Span}\{y_i \in C^\infty(T, R)\colon i = 1, \ldots, p\} = p$$
$$\dim \operatorname{Span}\{\varphi_j(y) \in C^\infty(T, R)\colon j = 1, \ldots, m\} = m$$
$$y_i \notin \operatorname{Span}\{\varphi_j(y)\colon j = 1, \ldots, m\}, \quad \forall i = 1, \ldots, p$$
$$\operatorname{col}(y_1(t), \ldots, y_p(t)) := y(t), \operatorname{col}(\varphi_1(y(t)), \ldots, \varphi_m(y(t))) := \varphi(y(t)) \tag{1.3}$$

constructive procedures are obtained for the solving the following problems.

Problem 1. It is required to determine the necessary and sufficient conditions for the differential realization problem to be solvable (the concurrent satisfaction of conditions *a* - *c*):

a) a finite dimensional phase manifold R^n of minimum dynamic order n exists (the so-called minimum state space),
b) a certain initial state $x(t_0) = x_0 \in R^n$ exists,
c) a differential system (1.1), (1.2) (with certain matrices A, B and C) exists which evolves (with an initial condition x_0) in the phase manifold R^n and has the same (identical) mapping $y = Cx$.

Problem 2. It is required to construct a direct realization algorithm for Problem 1: the calculation of the minimum dynamic order n, the vector $x_0 \in R^n$ and the matrices $A \in M_{\mathrm{n,n}}(R)$, $B \in M_{n,m}(R)$, $C \in M_{p,n}(R)$. In the general case, the constructions x_0, A, B and C are not unique.

The first equation of (1.3) denotes that any variable y_i, observed (experimentally) in a time interval T of the vector output signal y, cannot be expressed in terms of (replaced by) a certain linear combination of the other y_j (this leads to the minimum dimensionality of the space of the output signals). An analogous structural property for the coordinates of the vector function $\varphi(y)$ follows from the second equation of (1.3). The third condition is specific and it states, in a definite sense in view of the constraint (1.2), that the realization must be with the state space, since it necessarily follows from it that $y \ne D\psi(x)$, $\forall D \in M_{p,m}(R)$. Point *a* enables us to call such a realization a minimum realization[2] and, in matrix terms, this point is equivalent to a position when the pair (C, A) is observable.

Suppose $z_1, \ldots, z_k \in C^\infty(T, R^q)$, then the determinant $\Gamma_q(z_1, \ldots, z_k)$ of the matrix

$$\left\|\begin{matrix} \langle z_1, z_1\rangle_{C^\infty} & \langle z_1, z_2\rangle_{C^\infty} & \ldots & \langle z_1, z_k\rangle_{C^\infty} \\ \langle z_2, z_1\rangle_{C^\infty} & \langle z_2, z_2\rangle_{C^\infty} & \ldots & \langle z_2, z_k\rangle_{C^\infty} \\ \ldots & \ldots & \ldots & \ldots \\ \langle z_k, z_1\rangle_{C^\infty} & \langle z_k, z_2\rangle_{C^\infty} & \ldots & \langle z_k, z_k\rangle_{C^\infty} \end{matrix}\right\|$$

forms the Gram determinant[6] for $z_1, \ldots, z_k$. If any principal minor in $\Gamma_q(z_1, \ldots, z_k)$ is equal to zero, then $\Gamma_q(z_1, \ldots, z_k) = 0$ and this observation is useful for computational purposes when analysing the linear dependence of fixed sets of vector functions from $C^\infty(T, R^q)$ as the following lemma confirms.

Lemma 1. The ordered pair

$$(y, \varphi(y)) \in C^\infty(T, R^p) \times C^\infty(T, R^m)$$

satisfies requirement (1.3) only when

$$\Gamma_1(y_1, \ldots, y_p) \ne 0, \quad \Gamma_1(y_i, \varphi_1(y), \ldots, \varphi_m(y)) \ne 0, \quad \forall i = 1, \ldots, p$$

2. Realization in the case of complete measurement of a state vector

In this Section, we consider the problem of realization on the assumption that, in the formulation of Problem 2,

$$n: = p \;\text{ и }\; C := E_n \in M_{n,n}(R)$$

(E_n is a unit matrix), that is, the realization is considered with a trivial output mapping $y = x$ (a similar formulation is of current interest for quite a broad class of real physical systems[7]).

The necessary and sufficient conditions for a solution of this realization problem to exist are determined in the following theorem. These conditions are constructive in the sense that, firstly, by virtue of Lemma 1, they attract the guarantee of the structural constraints (1.3) and, secondly, an algorithm for the parametric construction of a realization model (see Problem 2) is built into them when the above mentioned conditions are confirmed for a specified evolution of the phase vector.

Theorem 1. Realization (1.1) exists (and, moreover, it is unique) for the pair

$$(x, \psi(x)) \in C^{\infty}(T, R^n) \times C^{\infty}(T, R^m)$$

if and only if the following equalities hold:

$$\Gamma_1(dx_i/dt, x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x))/\Gamma_1(x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x)) = 0, \quad \forall i = 1, \ldots, n$$

$$\operatorname{col}(x_1(t), \ldots, x_n(t)) := x(t), \operatorname{col}(\psi_1(x(t)), \ldots, \psi_m(x(t))) := \psi(x(t)) \tag{2.1}$$

In this case, the algorithm for identifying the matrices A and B of the differential system (1.1) in this realization is made up of the following matrix relation

$$[A, B] = \int_T \omega_d(\tau)[\omega(\tau)]^* d\tau \times \left[\int_T \omega(\tau)[\omega(\tau)]^* d\tau\right]^{-1}$$

$$\omega_d(t) := \operatorname{col}(dx_1(t)/dt, \ldots, dx_n(t)/dt) \in R^n,$$

$$\omega(t) := \operatorname{col}(x_1(t), \ldots, x_n(t), \psi_1(x(t)), \ldots, \psi_m(x(t))) \in R^{n+m} \tag{2.2}$$

where $[A, B]$ is a partitioned $n \times (n+m)$ matrix and $[\cdot]^*$ is the operation of transposition of a matrix-column. Hence, formula (2.2) enables us to establish the matrices A and B of system (1.1) using a posteriori data on the trajectory of the system $x \in C^{\infty}(T, R^n)$.

Remark.

1° A geometrical interpretation of condition (2.1) can be given

$$dx_i/dt = \upsilon(x_i) + h(x_i), \quad \upsilon(x_i) \in \operatorname{Span}(x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x)) \perp h(x_i)$$

$$\Gamma_1(dx_i/dt, x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x))/\Gamma_1(x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x)) = \langle h(x_i), h(x_i)\rangle_{C^{\infty}}$$

2° A solution of Eq. (2.2) exists since

$$\det\left[\int_T \omega(\tau)[\omega(\tau)]^* d\tau\right] = \Gamma_1(x_1, \ldots, x_n, \psi_1(x), \ldots, \psi_m(x)) \neq 0$$

3° It can be shown that, under conditions of the approximate modelling of the equations of state (1.1), relation (2.2) is a solution, by the method of least squares, of the problem of parametric optimization (for the elements of the matrices A and B) of the form

$$\min\langle dx/dt - Ax - B\psi(x), dx/dt - Ax - B\psi(x)\rangle_{C^{\infty}}$$

Example 1. Suppose the object being investigated is a regid body, the rotational motion of which is described by the Euler's equations

$$J_x d\omega_x(t)/dt + (J_z - J_y)\omega_y(t)\omega_z(t) = m_x(\omega_x(t)) \; (x, y, z) \tag{2.3}$$

where J_x, J_y, J_z are the moments of inertia about the principal axes, ω_x, ω_y, ω_z are the components of the angular velocity about these axes and m_x, m_y, m_z are the control moments, to represent of which we adopt the agreement that the linear damping

$$m_x(\omega_x(t)) = -k_x\omega_x(t) \; (x, y, z) \tag{2.4}$$

is organized with the aim of quenching the initial angular velocities.

Henceforth, the symbol (x, y, z) denotes that the two relations which have not been written out are obtained by circular permutation of the above-mentioned indices.

Remark. We now consider differential system (2.3) from the following "utilitarian" point of view (which is equivalent to solving the realization problem). Suppose a certain motion

$$\mathrm{col}(\upsilon_x(\cdot), \upsilon_y(\cdot), \upsilon_z(\cdot)) \in C^\infty([t_0, t_1], R^n)$$

is known in the time interval $[t_0, t_1]$. It is required to elucidate in which cases this motion is realizable in principle using a certain system (2.3), that is,

$$\mathrm{col}(\upsilon_x(\cdot), \upsilon_y(\cdot), \upsilon_z(\cdot)) = \mathrm{col}(\omega_x(\cdot), \omega_y(\cdot), \omega_z(\cdot))$$

Introducing the notation

$$x_1(t) := \omega_x(t), \quad a_1 = -k_x/J_x, \quad b_1 = (J_z - J_y)/J_x\ (1, 2, 3; x, y, z)$$

we reduce Eqs. (2.3) and (2.4) to the vector-matrix form (1.1):

$$\mathrm{col}(dx_1(t)/dt, dx_2(t)/dt, dx_3(t)/dt) = \mathrm{diag}(a_1, a_2, a_3)\mathrm{col}(x_1(t), x_2(t), x_3(t)) +$$
$$+ \mathrm{diag}(b_1, b_2, b_3)\mathrm{col}(x_2(t)x_3(t), x_1(t)x_3(t), x_1(t)x_2(t)) \tag{2.5}$$

Next, suppose the object possesses physical parameters equal to

$$J_x = 0.5, \quad J_y = 0.4, \quad J_z = 0.1, \quad k_x = k_y = k_z = 1$$

In this formulation, the "reference" coefficients (corresponding to the dynamics (2.3), (2.4)) of the differential model (2.5) take the values

$$a_1 = -2, \quad a_2 = -2.5, \quad a_3 = -10, \quad b_1 = -0.6, \quad b_2 = 1, \quad b_3 = -1$$

We now construct (by means of a numerical experiment using MATLAB[8]) the simulated motion of the angular velocity vector col($\omega_x(\cdot)$, $\omega_y(\cdot)$, $\omega_z(\cdot)$) of the object (2.3), (2.4) in the time interval $[t_0, t_1] = [0,1]$. To do this, we numerically integrate differential system (2.5) with the initial conditions

$$x_1(t_0) = 10, \quad x_2(t_0) = -20, \quad x_3(t_0) = 30$$

Now, for structural-parametric identification purposes, we use (for the non-linear component of the equations of state (1.1)) the structural hypothesis

$$\mathrm{col}(\psi_1(x), \psi_2(x), \psi_3(x)) := \mathrm{col}(x_2x_3, x_1x_3, x_1x_2) \tag{2.6}$$

which (using a simulated evolution of the angular velocity vector) imparts the following numerical values (δ_{3j} is the Kronecker delta) to constructions (2.1) and (2.2)

$$\Gamma_1(dx_j/dt, x_1, x_2, x_3, \psi_1(x), \psi_2(x), \psi_3(x))/\Gamma_1(x_1, x_2, x_3, \psi_1(x), \psi_2(x), \psi_3(x)) = 0.01\delta_{3j},$$
$$j = 1, 2, 3 \tag{2.7}$$

$$A = \begin{Vmatrix} -1.99 & 0.04 & 0.01 \\ -0.00 & -2.50 & 0.00 \\ -0.00 & -0.01 & -9.99 \end{Vmatrix}, \quad B = \begin{Vmatrix} -0.59 & -0.00 & -0.00 \\ 0.00 & 0.99 & 0.00 \\ -0.00 & -0.00 & -0.99 \end{Vmatrix} \tag{2.8}$$

The motions of the *reference* and *identified* models are represented in the upper part of Fig. 1. Here, ω_x, ω_y, ω_z are the components of the angular velocity of the reference model and w_x, w_y, w_z are the components of the angular velocity of the identified model. The current mismatch between the angular velocities of the reference and identified models

$$\Delta\omega_x(t) = \omega_x(t) - w_x(t)\ (x, y, z)$$

is shown in the lower part of Fig. 1.

Analysis of the numerical results (2.7) and (2.8) (including the graphs in Fig. 1) together with Theorem 1 shows that, without (hypothetically) possessing Euler's equations (2.3), it is, in essence, possible to "establish" them empirically with an a priori assumption on the form of the structural law (2.6).

3. Realization of a differential system in the case of incomplete measurement of the state vector

Endevouring to represent the solution of the realization problem in terms of the subspaces of the space $C^\infty(T, R^p)$, we will now consider some formal constructions, devoting most of our attention to the geometrical content of the theorems introduced. We will start from a definition, the mathematical basis of which is the structure of a modular lattice.[9]

Definition 1. We shall call a finite sequence $\langle L_j\rangle_{j=0,\ldots,k}$ of sets $L_j \subset C^\infty(T, R^P)$ an $\langle L_j\rangle_k$-cortege whenever the relations

$$L_0 \subset L_1 \subset \ldots \subset L_{k-1} \subset L_k, \quad L_0 \neq L_1 \neq \ldots \neq L_{k-1} \neq L_k$$

hold.

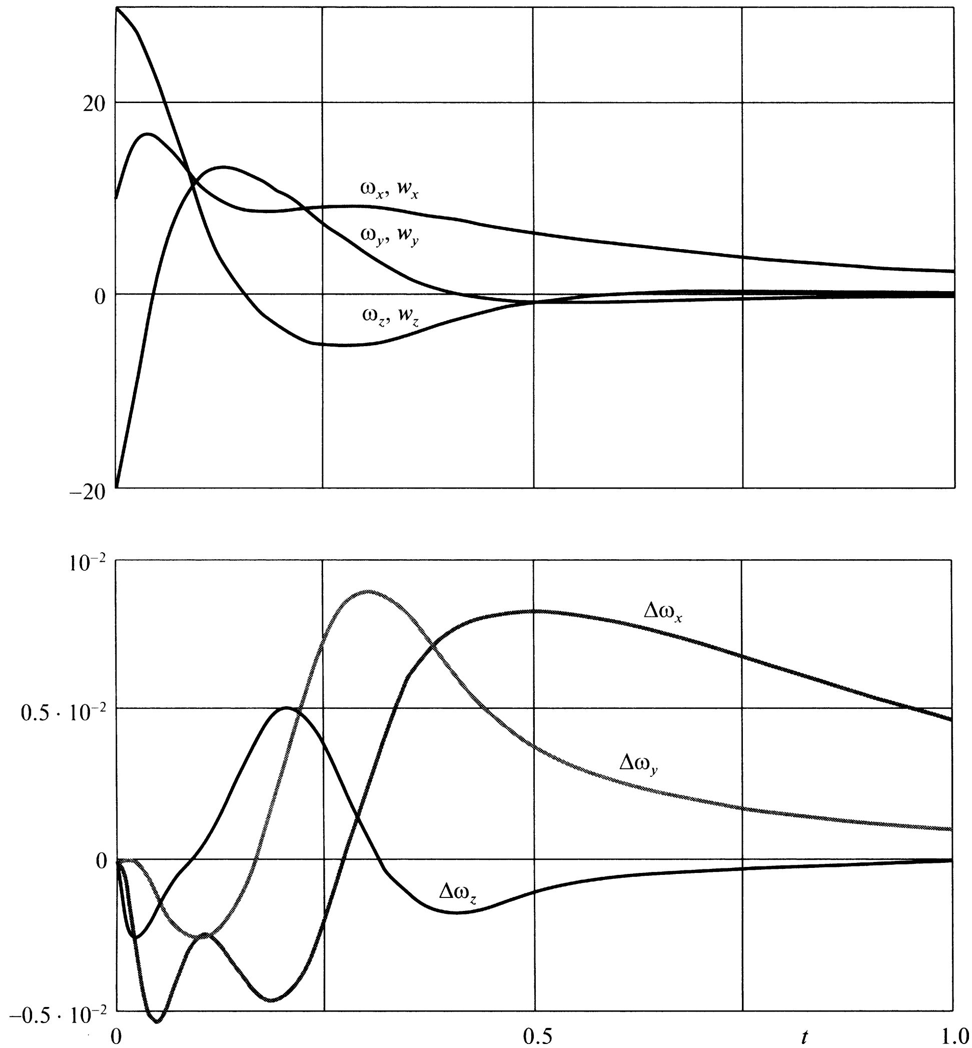

Fig. 1.

At the same time, we shall say that a $\langle L_j\rangle_k$-cortege possesses an index l if

$$\operatorname{Span} L_k = \operatorname{Span} L_{k-1}, \quad \dim \operatorname{Span} L_{k-1} = l$$

The property "an $\langle L_j\rangle_k$-cortege possesses an index l" can be established by a numerical procedure, the basic elements of which are contained in Lemma 2.

Lemma 2. In order that a $\langle L_j\rangle_k$-cortege has an index l, it is necessary and sufficient that l functions $z_1, \ldots, z_1 \in L_{k-1}$ should be found such that $\Gamma_p(z_1, \ldots, z_1) \neq 0$ while, for any of the $l+1$ functions from L_k, the Gram determinant should be equal to zero.

Suppose $(y, \varphi(y)) \in C^\infty(T, R^p) \times C^\infty(T, R^m)$, $\{e_1, \ldots, e_p\}$ is a standard basis in R^p (in which the i-th component of the vector e_i is equal to 1 and the remaining components are equal to zero) and $y^{(k)} := d^k y/dt^k$, $\varphi^{(i)}(y) := d^i\varphi(y)/dt^i$ (where $y^{(0)} = y$, $\varphi^{(0)}(y) = \varphi(y)$), and suppose

$$U_{ij} := \{(\varphi_j^{(i)}(y)e_1, \ldots, \varphi_j^{(i)}(y)e_p\}, \quad j = 1, \ldots, m$$

Definition 2. For the pair $(y, \varphi(y)) \in C^\infty(T, R^p) \times C^\infty(T, R^m)$ and non-negative integers k and q, we call a set $S_{k,\,qpm} \subset C^\infty(T, R^p)$ of the form

$$S_{k,qpm} := \{y^{(0)}, y^{(1)}, \ldots, y^{(k)}\} \cup \{U_{ij}\colon i = 0, \ldots, q; j = 1, \ldots, m\}$$

the k, qpm-stream of the vector function $(y, \varphi(y))$ in the time interval T.

If a vector $b \in R^n$ is found for the matrix $A \in M_{n,n}(R)$ such that

$$\operatorname{Span}\{b, Ab, A^2b, \ldots, A^{n-1}b\} = R^n$$

then a matrix A is called (Ref. 10, p.262) a cyclic matrix, and b is its cyclic generator. A matrix A is cyclic if and only if its characteristic polynomial is equal to the minimum polynomial[9] and, at the same time, the pair (A, b) is completely controllable (Ref. 2, Theorem 3.4) or, equivalently, when precisely one Jordan block corresponds to each eigenvalue of a matrix A and, also, when a matrix A can be reduced to the Frobenius form (Ref. 10, p. 263, Lemma P.8)

The realization procedure is closely connected with the construction of a matrix X which is established by the following lemma.

Lemma 3. Suppose $A \in M_{n,n}(R)$ is a cyclic matrix, $b \in R^n$ is its cyclic generator and $c \neq 0 \in R^n$. A unique matrix $X \in M_{n,n}(R)$ then exists such that

$$XA = AX, \quad Xb = c$$

Proof. The vector system $\{b, Ab, A^2b, \ldots, A^{n-1}b\}$ forms a basis in R^n and, consequently, the vector c has a unique representation of the form

$$\beta_0 b + \beta_1 Ab + \beta_2 A^2 b + \ldots + \beta_{n-1}A^{n-1}b = c \quad \square$$

Now (when account is taken of the equality $Xb = c$) the construction of the matrix X is completely obvious:

$$X = \beta_0 E_n + \beta_1 A + \beta_2 A^2 + \ldots + \beta_{n-1}A^{n-1}$$

The coefficients β_i are the solution of the linear system

$$\begin{Vmatrix} \langle z_0, z_0\rangle_{R^n} & \langle z_0, z_1\rangle_{R^n} & \ldots & \langle z_0, z_{n-1}\rangle_{R^n} \\ \langle z_1, z_0\rangle_{R^n} & \langle z_1, z_1\rangle_{R^n} & \ldots & \langle z_1, z_{n-1}\rangle_{R^n} \\ \ldots & \ldots & \ldots & \ldots \\ \langle z_{n-1}, z_0\rangle_{R^n} & \langle z_{n-1}, z_1\rangle_{R^n} & \ldots & \langle z_{n-1}, z_{n-1}\rangle_{R^n} \end{Vmatrix} \begin{Vmatrix} \beta_0 \\ \beta_1 \\ \ldots \\ \beta_{n-1} \end{Vmatrix} = \begin{Vmatrix} \xi_0 \\ \xi_1 \\ \ldots \\ \xi_{n-1} \end{Vmatrix}$$

Here, $z_i := A^i b$ (we assume that $A^0 := E_n$) and ($i = 0, \ldots, n-1$). Since the matrix A is cyclic and b is its cyclic generator, the Gram determinant (with respect to z_i) is a non-zero determinant and, consequently, the system has a unique solution with regard to β_i. Finally, bearing in mind the fact[2] that, for a cyclic matrix A, the family of all the matrices D commuting with it possesses the structure $\{D \in M_{n,n}(R): D = \eta(A), \eta(\lambda) \in \Xi\}$, where Ξ is a class of polynomials of the variable λ of a degree not exceeding n, we conclude that the matrix X is unique.

Remark. It is possible to give a somewhat different definition: matrices which commute with the matrix A have the form $\eta(A)$, where $\eta(A)$ is a polynomial in the ring of classes of residues with respect to the modulus of the minimum polynomial of the matrix A.

Lemma 4. Suppose $x_0 \in R^n$ and the matrix $D \in M_{n,n}(R)$ commutes with the matrix $A \in Mn,n(R)$. Then, $z(t) = Dx(t)$, $t \in T$, if $z: T \to R^n$ and $x: T \to R^n$ are solutions of the equations

$$dz(t)/dt = Az(t), \quad z(t_0) = Dx_0 \in R^n, \quad dx(t)/dt = Ax(t), \quad x(t_0) = x_0 \in R^n$$

Before presenting the solution of the problem of the realization of a non-linear autonomous system with the hypothesis (1.2), we will describe it for the case of a homogeneous linear system.

Theorem 2. An a posteriori process $y \in C^\infty(T, R^p)$ (1.3) satisfies the realization (1.1) of the minimum dynamic order n with a law $\Psi(x) \equiv 0$ if and only if the $\langle S_j\rangle_n$-cortege of the j-jet (j = 0, ..., n) of the function y in T has an index n.

Proof. *Necessity.* Suppose y admits of a linear realization of the minimum order n. We differentiate the second equation of equality (1.1) n times and substitute $dx(t)//dt$ from the first equation (when $\Psi(x) \equiv 0$). We obtain

$$y^{(0)}(t) = Cx(t), \quad y^{(1)}(t) = CAx(t), \ldots, y^{(n)}(t) = CA^n x(t)$$

We now add them, having multiplied the first equality by a_0 and the second by a_1 and so on, and the last by unity, where a_i ($i = 0, \ldots, n-1$) are the coefficients of the normalized characteristic polynomial $\chi(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ of the matrix A. According to the Hamilton-Cayley theorem, the matrix A satisfies the characteristic equation $\chi(A) = 0$. As a result, we arrive at the differential equation

$$y^{(n)}(t) + \sum_{i = 0, \ldots, n-1} a_i y^{(i)}(t) = 0 \tag{3.1}$$

from which it follows that, using measurements of the jet S_n in the time interval T, the coefficients $a_i (i = 0, \ldots, n-1)$ can be calculated from the system

$$\begin{Vmatrix} \langle z_1, z_1\rangle_{C^\infty} & \langle z_1, z_2\rangle_{C^\infty} & \ldots & \langle z_1, z_n\rangle_{C^\infty} \\ \langle z_2, z_1\rangle_{C^\infty} & \langle z_2, z_2\rangle_{C^\infty} & \ldots & \langle z_2, z_n\rangle_{C^\infty} \\ \ldots & \ldots & \ldots & \ldots \\ \langle z_n, z_1\rangle_{C^\infty} & \langle z_n, z_2\rangle_{C^\infty} & \ldots & \langle z_n, z_n\rangle_{C^\infty} \end{Vmatrix} \begin{Vmatrix} a_{n-1} \\ a_{n-2} \\ \ldots \\ a_0 \end{Vmatrix} = \begin{Vmatrix} \xi_1 \\ \xi_2 \\ \ldots \\ \xi_n \end{Vmatrix}$$

$$z_i = y^{(n-i)}, \quad \xi_i = -\langle y^{(n)}, y^{(n-i)}\rangle_{C^\infty}, \quad i = 1, \ldots, n \tag{3.2}$$

According to Eq. (3.1), the set S_n depends linearly on $C^\infty(T, R^p)$. We will now show that the dimension of the linear span, Span S_n, is equal to n and its algebraic basis forms the (n - 1)-jet of the function y.

We will argue the opposite. Suppose n real θ_i are found (not all of which are equal to zero) such that

$$\sum_{i=0,\ldots,n-1} \hat{a}_i y^{(i)}(t) = 0$$

Without loss of generality, it can be assumed that $\hat{a}_{n-1} = 1$ (otherwise, the order of the last equation will be lower than $n-1$). This assumption (the last equality) leads to a contradiction with regard to the conditions (assumptions) made above concerning a minimum order equal to n in the case of the realization system, since, in this case, a differential realization exists for y with a minimum dimension equal to $n-1$ (the dimension of the state space), for example,

$$dz(t)/dt = \hat{A}z(t), \quad t \in T, \quad y(t) = \hat{C}z(t)$$

$$\hat{A} = \begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -\hat{a}_0 & -\hat{a}_1 & -\hat{a}_2 & \ldots & -\hat{a}_{n-2} \end{Vmatrix}$$

The construction of the matrix is considered below; we will confine ourselves to the remark that the matrix depends on the vector $z(t_0)$, the matrix B and the choice of the cyclic generator for B (see the structure of the matrix C in relations (3.5)).

Sufficiency. Suppose the $\langle S_j \rangle_n$-cortege of the left of the function y has an index n in the time interval T. Since the set S_n is linearly dependent and dim Span S_{n-1}, a unique set of real numbers $a_i (i=0, \ldots, n-1)$ is found such that

$$y^{(n)}(t) + \sum_{i=0,\ldots,n-1} a_i y^{(i)}(t) = 0$$

Now, denoting the j-th coordinate $(j=1, \ldots, p)$ of the vector function y by y_i, we introduce the auxiliary state vector $\hat{y}_j \in C^\infty(T, R^n)$ with the variables $\hat{y}_{ji}$:

$$\hat{y}_{ji}(t) := y_j^{(i-1)}(t), \quad i = 1, \ldots, n \tag{3.3}$$

As a result, we arrive at the differential vector-matrix system

$$d\hat{y}_j(t)/dt = A\hat{y}_j(t), \quad t \in T, \quad y_i(t) = \hat{c}\hat{y}_j(t), \quad \hat{c} = [\mathrm{col}(1, 0, \ldots, 0)]^*$$

Here, the matrix A differs from the matrix $\hat{A}$ in the last row which has the form

$$\| -a_0 \; -a_1 \; -a_2 \ldots -a_{n-1} \|$$

Suppose b is a certain (Ref. 9, Lemma P.10) cyclic generator of the matrix A and suppose $D_{ji} \in M_{n,n}(R)$ is a matrix which satisfies the algebraic conditions

$$D_j A = AD_j, \quad D_j b = \hat{y}_j(t_0)$$

Such a matrix D_j exists by virtue of Lemma 3, and

$$D_j = \beta_0 E_n + \beta_1 A + \beta_2 A^2 + \ldots + \beta_{n-1} A^{n-1} \tag{3.4}$$

where the coefficients β_i $(i=0, \ldots, n-1)$ are the solution of the linear system

$$\beta_0 b + \beta_1 Ab + \beta_2 A^2 b + \ldots + \beta_{n-1} A^{n-1} b = \hat{y}_j(t_0)$$

We now introduce the state vector $x(t)$ of the system $dx(t)/dt = Ax(t)$ into the treatment. Then, $\hat{y}_j(t) = D_j x(t)$, $x(t_0) = b$ which holds by virtue of Lemma 4. As a results, we arrive at the realization system for the individual variable y_j of the output signal

$$dx(t)/dt = Ax(t), \quad x(t_0) = b, \quad t \in T, \quad y_j(t) = \hat{c}D_j x(t).$$

A similar realization (with the same A, b and $\hat{c}$ but its own D_j) can be constructed for any $j=1, \ldots p$. Hence, the linear homogeneous realization of y has the form

$$dx(t)/dt = Ax(t), \quad x_0 = b, \quad t \in T, \quad y(t) = Cx(t)$$

$$C = \mathrm{col}(\hat{c}D_1, \ldots, \hat{c}D_p) \tag{3.5}$$

Corollary 1. The process $y \in C^\infty(T, R^p)$ has the realization (1.1) of minimum dynamic order n with $\Psi(x) \equiv 0$ if and only if

$$\Gamma_p(\hat{s}_1, \ldots, \hat{s}_{n+1})/\Gamma_p(s_1, \ldots, s_n) = 0, \quad \hat{s}_1, \ldots, \hat{s}_{n+1} \in S_n, \quad s_1, \ldots, s_n \in S_{n-1}$$

where S_n and S_{n-1} are the n-jet and the $(n-1)$-jet of the vector function y, respectively, and, at the same time, the matrices A, B and C of system (1.1) satisfy relations (3.2)–(3.5).

Corollary 2. The pair (C, A) is observable and the matrix A is cyclic (or, what is equivalent, the normalized minimum polynomial of the matrix A is equal to its normalized characteristic polynomial).

The calculations are illustrated on the basis of given trajectory measurements which explain the method of describing both the minimum dynamic order and the characteristic polynomial of the matrix A on the methodological basis of Corollary 1.

Example 2. We now consider the construction of Example 1 in a linearized formulation, that is, when $B=0$. It is clear that, in the case of the "reference" model, the minimum dynamic order will have an index $n=3$ and the characteristic polynomial of the matrix A is equal to

$$\chi(\lambda)=\lambda^3+14.5\lambda^2+50\lambda+50 \tag{3.6}$$

The result of the identification is shown below for two versions ($p=1$ and $p=3$) of the model of the measuring device (the simulated model of the motion was calculated as in Example 1):

$$y(t):=\omega_x(t)+\omega_y(t)+\omega_z(t)\ (p=1),\quad y(t):=\mathrm{col}(\omega_x(t),\omega_y(t),\omega_z(t))\ (p=3)$$

$$\Gamma_p(y^{(0)},y^{(1)})/\Gamma_p(y^{(0)})=6.076\cdot 10^2\ (p=1),\ \ldots=1.827\cdot 10^3\ (p=3)$$

$$\Gamma_p(y^{(0)},y^{(1)},y^{(2)})/\Gamma_p(y^{(0)},y^{(1)})=1.029\cdot 10\ (p=1),\ \ldots=2.900\cdot 10^2\ (p=3)$$

$$\Gamma_p(y^{(0)},y^{(1)},y^{(2)},y^{(3)})/\Gamma_p(y^{(0)},y^{(1)},y^{(2)})=5.638\cdot 10^{-9}\ (p=1),\ \ldots=-1.064\cdot 10^{-8}\ (p=3)$$

$$\chi^*(\lambda)=\lambda^3+14.5\lambda^2+50\lambda+50 - \text{ is the identified polynomial}\quad (p=1,p=3)$$

The characteristic polynomial $\chi^*(\lambda)$ identified (according to equality (3.2)) has the form of the "reference" polynomial (3.6) and results from the application of Theorem 2.

Proof of the following theorem is the step which generalizes the results of Theorem 2.

Theorem 3. The pair $(y,\varphi(y))\in C^\infty(T,R^p)\times C^\infty(T,R^m)$ has a minimum realization (1.1)–(1.3) of order n if and only if a minimum with respect to the k $\langle S_{j,(k-1)pm}\rangle_k$-cortege of the j,(k -1)pm-jet, $j=0$, ..., k, is found for $(y,\varphi(y))$ in T possessing the structural index l such that $l=k+pm$, $k=1$ and $k+pm\le l\le(k-1)(1+pm)+1$, $k\ge 2$, and, at the same time, $kp\ge n\ge k$.

Proof *(. Sufficiency).* Since

$$\mathrm{Span}\, S_{k,(k-1)pm}=\mathrm{Span}\, S_{(k-1),(k-1)pm}$$

a certain set of real numbers $\hat{a}_i (i=0,1,\ldots,k-1)$ and a complete set of matrices $G_j\in M_{p,m}(R)$ ($j=0,1,\ldots,k$-1) are found such that the vector function $(y,\varphi(y))$ will satisfy the vector-matrix differential equation

$$y^{(k)}(t)+\sum_{i=0,\ldots,k-1}\hat{a}_i y^{(i)}(t)=\sum_{i=0,\ldots,k-1}G_j\varphi^{(j)}(y(t)) \tag{3.7}$$

We now outline the realization procedure (while continuing to make headway with the proof). It can be conditionally divided up into two steps: in Step 1, we shall construct a realization of Eq. (3.7) in terms of a certain kp-dimensional system (consequently, $n\le kp$) and, in Step 2, the system of minimum dynamic order is constructed. Speaking more formally, in Step 2, a "factor-system" must be constructed using the modulus of the maximum "non-observable subspace" of the space of the states of the realization system found in Step 1.

Step 1 We now introduce $z(t)$, a kp-dimensional state vector, into the treatment

$$z(t):=\mathrm{col}(z_1(t),z_2(t),\ldots,z_k(t))\in R^{kp}$$

$$z_1(t)=y^{(0)}(t),\quad z_2(t)=y^{(1)}(t),\ \ldots,\ z_k(t)=y^{(k-1)}(t)$$

This representation (by virtue of differential equation (3.7)), leads to the new vector-matrix differential system

$$dz(t)/dt = \hat{A}\, z(t) + \hat{H}\,\varphi(y(t)), \quad t \in T$$

$$z_0 = z(t_0) = \mathrm{col}(y^{(0)}(t), y^{(1)}(t), \ldots, y^{(k-1)}(t))\big|_{t = t_0}$$

$$y(t) = \hat{C}\, z(t)$$

$$\hat{A} = \begin{Vmatrix} 0 & E_k & 0 & \ldots & 0 \\ 0 & 0 & E_k & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & E_k \\ -\hat{a}_0 E_k & -\hat{a}_1 E_k & -\hat{a}_2 E_k & \ldots & -\hat{a}_{k-1} E_k \end{Vmatrix}$$

$$\hat{C} = \|E_k, 0, \ldots, 0\|,\ \hat{C}\,(\hat{A}^{k-j-1} + \hat{a}_{k-1}\hat{A}^{k-j-2} + \ldots + \hat{a}_{j+2}\hat{A} + \hat{a}_{j+1} E_{kp})\hat{H} = G_j$$

$$j = 0, \ldots, k-1$$

$$\hat{A} \in M_{kp,kp}(R), \quad \hat{H} \in M_{kp,m}(R),\ \hat{C} \in M_{p,kp}(R) \tag{3.8}$$

The realization problem is solved within the limits of system (3.8), though, possibly, with a greater generality than is ordinarily necessary. As a rule, its dynamic order does not correspond to the minimality condition which is important in overcoming the "overdependence" of the realization problem. The second step in the realization removes this difficulty.

Step 2 consists of the reduction of differential system (3.8) to the *minimum* realization. This is done using the construction of the maximum non-observable subspace $N \subset R^{kp}$ of system (3.8) (Ref. 9):

$$N = \cap\{\mathrm{Ker}\,(\hat{C}, \hat{A}^{i-1}) : i = 1, \ldots, kp\} \neq 0 \in R^{kp}$$

which represents the greatest $\hat{A}$ -invariant subspace contained in the subspace $\mathrm{Ker}\hat{C}$ (the kernel of the matrix $\hat{C}$).

Suppose $n = kp$ - dim N. We denote by R^n the factor-space R^{kp}/N and by $\hat{A}$ the linear mapping induced in R^n by $R^n \to R^n$. By virtue of the inclusion $\mathrm{Ker}\hat{C} \supset N$, the linear mappings B: $R^m \to R^p$ and C: $R^n \to R^p$ exist such that

$$B = P\hat{H}, \quad CP = \hat{C}$$

where P: $R^{kp} \to R^n$ is a canonical factor-mapping with respect to the modulus N. Hence, we have the possibility of constructing a factor-system of the realization of the form

$$dx(t)/dt = Ax(t) + B\varphi(y(t)), \quad x(t_0) = Pz_0, \quad t \in T$$

$$y(t) = Cx(t) \tag{3.9}$$

Since $\dim R^n \leq \dim R^{kp}$, the order n of system (3.9) does not exceed the magnitude of kp.

Necessity. Suppose $(y, \varphi(y))$ satisfies system (1.1) on the minimum dimension n and suppose k is the degree of the minimum polynomial of the matrix A. Differentiating the second equation of the system k times and, each time, substituting the expression for $dx(t)/dt$ from the first equation, we obtain as a result the $k+1$ equalities

$$y^{(0)}(t) = Cx(t), \quad y^{(1)}(t) = CAx(t) + CB\varphi^{(0)}(y(t)), \ldots, y^{(k)} = CA^k x(t) +$$

$$+CA^{k-1}B\varphi^{(0)}(y(t)) + CA^{k-2}B\varphi^{(1)}(y(t)) + \ldots + CB\varphi^{(k-1)}(y(t)) \tag{3.10}$$

We add these equalities, having multiplied the first by a_0, the second by a_1 and so on, and the last by one, where a_i are the coefficients of the normalized minimum polynomial

$$\pi(\lambda) := \lambda^k + a_{n-1}\lambda^{k-1} + \ldots + a_1\lambda + a_0$$

of the matrix A. Taking account of the fact that $\pi(A) = 0$, we conclude that the term $x(t)$ on the right-hand side of the sum disappears. As a result, we arrive at the differential equation

$$y^{(k)}(t) + \sum_{i = 0, \ldots, k-1} a_i y^{(i)}(t) = \sum_{i = 0, \ldots, k-1} G_j \varphi^{(j)}(y(t)) \tag{3.11}$$

By virtue of equalities (3.10), the matrices $G_j \in M_{p,n}(R)$ have the form

$$G_j = C(A^{k-j-1} + a_{k-1}A^{k-j-2} + \ldots + a_{j+2}A + a_{j+1}E_n)B, \quad j = 0, \ldots, k-1$$

By virtue of differential equation (3.11), firstly, the equality

$$\text{Span } S_{k,(k-1)pm} = \text{Span } S_{(k-1),(k-1)pm}$$

is satisfied for the corresponding streams and, secondly, since the degree of the polynomial $\pi(\lambda)$ is no greater than n, the minimum dynamic order of the realization (1.1) complies with the estimate $n \geq k$. Hence, if

$$\dim \text{Span } S_{k-1,(k-1)pm} = (k-1)(1+pm)+1$$

then the proof can be considered as completed. Otherwise, the smallest integer j ($k-1 > j \geq 1$) is obviously found such that

$$j + pm \leq \dim \text{Span } S_{j,(k-1)pm} = \dim \text{Span } S_{j-1,(k-1)pm} = l \leq j + (k-1)pm$$

It is clear that the index j is smaller than the minimum dynamic order n and, at the same time, $n \leq jp$. The version $n > jp$ leads to a contradiction since a realization can always be constructed for $(y, \varphi(y))$ which is analogous to representation (3.8).

Corollary 3. The pair (C, A) of the minimum realization (3.8) is observable and the degree of the smallest polynomial of the matrix A is equal to k.

Remark. In the general case, the matrix A of system (3.9) is not cyclic (cf. Corollary 2) and it therefore does not always possess a canonical Frobenius form or, what is equivalent, no one Jordan cell corresponds to a certain (possibly, non-unique) eigenvalue of matrix A (see Example 4 below).

The cases when the inequalities in Theorem 3 have the form of equalities are illustrated by Example 3 ($l = k + pm$, $n = k$) and Example 4 ($l = k + pm$, $n = kp$).

Example 3. Suppose

$$y(t) = \text{col}(e^{2t}, e^t - 1), \quad \varphi(y(t)) = 1, \quad t \in T = [0, 1]$$

In this formulation $(y, \varphi(y))$ is the process (1.3) with the indices $p = 2$ and $m = 1$. Theorem 3 holds with $k = 2$ and $l = 4$, and Eq. (3.7) reduces to the differential form

$$y^{(2)} - 3y^{(1)} + 2y = -2\varphi(y)e_2$$

It is clear that the realization of the minimum dynamic order $n = k$ in a Jordan basis (in this case the realization matrix is cyclic) has the form

$$\begin{aligned} &dx_1(t)/dt = 2x_1(t), \quad dx_2(t)/dt = x_2(t) + \varphi(y(t)), \quad t \in T; \quad x_1(0) = 1, \quad x_2(0) = 0 \\ &y_1(t) = x_1(t), \quad y_2(t) = x_2(t) \end{aligned} \tag{3.12}$$

Example 4. Suppose

$$y(t) = \text{col}(e^t, e^t + e^{2t}), \quad \varphi(y(t)) = y_1^2(t), \quad t \in T = [0, 1]$$

where $(y, \varphi(y))$ is the process (1.3) with $p = 3$ and $m = 1$. Theorem 3 is applicable with $k = 1$, $l = 3$, and Eq. (3.7) reduces to the form

$$y^{(1)} - y = \varphi(y)e_2$$

The minimum realization (with $n = kp$) has a canonical Jordan form and differs from the construction (3.2) in the replacement of the right-hand side of the first equation by $x_1(t)$ and the condition $x_2(0) = 0$ by $x_2(0) = 2$. In this case, the realization matrix is not cyclic, indicating (see Corollary 2) that the order ($n = 2$ in the given case) of the "input - state - output" (the "input" = $\varphi(y)$) system cannot in all cases be recovered using data from measurements obtained for the *characteristic motion* of the object, that is, by determining the minimum order of the homogeneous system ($n = 1$) beforehand (which, for example, was cultivated[5] in calculating the order of a system with a control).

4. Conclusion

The approach proposed in this paper includes the treatment, on a rigorous mathematical basis, of the methodology involved in the differential modelling of multidimensional autonomous systems which describe (in the smallest state space) the dynamics of a posteriori processes[11] and, at the same time, pursue both general and specific aims:

- to develop a formal compact language of the jet corteges in which it is possible to discuss questions of the exact mathematical modelling of the invariants of autonomous linear and non-linear multidimensional differential realizations with a structural constraint (1.2) (the modelling of the minimum dynamic order of a realization, its characteristic polynomial, etc.);
- to propose direct algorithms for the structural-parametric identification of autonomous systems of minimum dynamic order. In particular, the proof of Theorem 2 and its Corollary 1 determine the basis algorithmic steps in the construction of a homogeneous system for the realization of the minimum order for the characteristic motion of a distributed object which admits a differential approximation[12] by the method of truncation of the number of harmonics:

Step 1-determination of the minimum order n of the homogeneous system of differential equations of the realization by calculating the corresponding index of the jet cortege of the process according to the characteristic criterion of Corollary 1;
Step 2-calculation of the elements $\{-a_0, -a_1, \ldots, -a_{n-1}\}$ of the Frobenius matrix A of the homogeneous system of differential equations which is modelled on the basis of the minimum order n and the solution of system (3.2) constructed in Step 1;
Step 3-fixing the initial state vector x_0 of the realization system in the form of a certain (any specified) cyclic generator b of the matrix A and calculation of the matrices $D_j(j = 1, \ldots, p)$ and C of the output signal using expressions (3.3)–(3.5) on the basis of the vector b and the phase position of the $(n - 1)$-jet at the instant t_0.

The ideas presented in this paper can result in the development of theoretical searches for the structural-parametric identification of complex dynamical systems in several different directions:

- for systems with a preset control, including a discontinuous control (in this context, we have in mind the natural symbiosis with the theory of distributions);
- for systems which are described by equations with delays;
- for systems in a separable Hilbert space[13] including time-dependent systems[14] with $t \to \psi(x(t))$ in the class of Bochner integrable functions.

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